## THE TEACHING OF MATHEMATICS.

In the teaching of natural science, the past quarter of a century has brought about a reform of the most sweeping nature. Methods of instruction have been radically changed, and the whole point of view from which the study of the sciences is approached has shifted. In its work in connection with chemistry, physics, natural history, botany, etc., secondary instruction has abandoned the abstract for the concrete. The student has now to deal, not so much with formula and law as with actual experiment—or as some enthusiasts would put it, with cause and effect. Of the good which has resulted from this change we need not speak. Of the danger which attends this method of instruction, of the points wherein it is inferior to the method it has supplanted, we have here no room for discussion. All that concerns us at the present moment is the fact that this change *has* taken place. Empiricism is the watchword of to-day. "Read nature in the language of experiment," cries the reformer. The cry has been heard and heeded; and the high school or academy, to say nothing of the college and scientific school, which is not well equipped with laboratory and apparatus, is not looked upon as "progressive," is not "up with the times."

In another department of instruction we see to-day the beginnings of a reform no less important, and certainly no less needed than was reform in the teaching of natural science. Ten years ago the teaching of English in our secondary schools was the merest farce. Latin and Greek, and in many cases French and German, received most liberal allowances of time and most painstaking care on the part of the teacher. But of systematic work in English there was none. Of comprehension on the part of the teacher that such work was needed there was almost none. The public apathy, yes, the apathy among educational leaders even, on the subject of instruction in the mother tongue was most disheartening. But the reform so sorely needed has at last begun. The tangible results that have as yet been produced are insignificant. The actual instruction given in English is meagre and often very poor. The public is still apathetic, and the great army of primary teachers is worse than apathetic. But the college and the secondary school have fairly begun the work of placing on a

sounder basis instruction in this most fundamental subject. Some echoes have reached us from across the water of the agitation in Germany which has for its object the elevation of the study of German; of the utterances of the leading French educators on the study of a nation's own language and literature; and of the revolutionary change in Sweden which has made the study of Swedish one of the corner stones of the national educational system of that country. These tidings from abroad, coupled with the earnest endeavors of our own educational leaders, have fairly inaugurated among us this most important reform. The study of our own language and literature is at last coming to receive the attention it deserves; and no one can for an instant doubt that the near future will see a leading place given in our schools to this long neglected subject, this branch of study which lies at the root of all true culture.

Science, language, mathematics. About these three branches may be grouped all that is most important in the purely mental part of the school training of children. In the teaching of science, reform has already come. In language it is at hand. In mathematics—well, what can be said of the present status of instruction in mathematics, and of the need and the possibility of reform in its methods? Few will dissent from the proposition that mathematics is one of the essentials of education. No one whose opinion is worthy of respect will deny that our schools, from the first steps of the primary grade to the close of the mathematical work in college, are in need of reform. But reform in the teaching of mathematics must, from the very nature of the case, be slower of growth and more difficult of achievement than has been the case in the natural sciences. About the subject of mathematics there is nothing to inspire the enthusiasm or to awaken the spontaneous interest that may be made to accompany work in natural science. Hence the reform, whenever or however it may come, will come slowly, reluctantly, and with uncertain step. But come it must, and come it will; nay, rather, come it shall.

Selecting for our consideration those branches of mathematics which properly belong to secondary education, *i. e.*, algebra, geometry, and trigonometry, let us examine briefly the reasons why mathematics must fairly be considered a difficult subject, much more difficult, for example, than natural science. In con-

nection with the latter the teacher can summon to his assistance a multitude of external aids. A difficult point can be illustrated and explained by experiment after experiment, each in itself sufficiently entertaining and valuable to hold the student's attention; each possessing at the same time the charm of being something that can be done by his own hands, and that can appeal directly to his mind through the medium of the five senses. He is brought to see and to comprehend clearly relations which in the abstract could be acquired only through long and difficult processes. His quick wit detects practical applications which may be useful to him in his future life; and he is made to feel in various ways that he is, in the study of science, acquiring something whose value he can comprehend. In a word, he is entertained and interested, and he sees the practical value of what he is doing.

What, on the other hand, is likely to be his mental attitude toward mathematics? He finds himself here brought face to face with, and expected to acquire, something essentially abstract and theoretical. Now and then his common sense does find some fact or principle that is really practical: certain laws respecting the computation of areas and volumes are established in his elementary geometry; the unknown quantities of algebra furnish an easy method of solving many practical problems with which the future may confront him; the sine, cosine, tangent, etc., of trigonometry are indispensable tools in ordinary surveying. But of what use is it, he asks, to know that the sum and the product of two conjugate imaginary numbers are both real numbers? Why is he bothered about the division of a line into its extreme and mean ratio? Who cares whether or not $\sin x + \sin y = 2 \sin \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y)$? He can imagine no possible use for these things, and his mind instinctively rebels against them. In a word, he is not entertained, he is not interested, and he does not see the practical value of the work upon which he is compelled to spend his time. The result of all this it is easy to see. The student who has no natural taste or aptitude for mathematics becomes disgusted, then angry, then disheartened. Hard work, induced by sincere love of his other studies or by fear of the disgrace of failure, may carry him through his prescribed course; but it is a wearisome experience, and it leaves on his mind a deep seated aversion for the work he has been through, and causes him throughout his future life to look back on his mathematics as the "worst grind" of his entire school and college life.

The facts here stated are incontrovertible. They are borne out by the experience or observation of almost every scholar who has ever pursued a college, or even a high school course, and they constitute one of the greatest blots on the American educational system of the past century. And what, meanwhile, has been the condition of mathematical *instruction*? If, as is undeniably true, mathematics is the most difficult branch in our commonly prescribed curricula, it would certainly seem as though its instruction should be entrusted only to those of most scholarly attainments and ripest preparation for their work. But the exact reverse of this always has been and is to-day true in this country. In the early part of this century it was almost invariably the case that instructors in mathematics were chosen from among those who had distinguished themselves in their classical studies. Forty or fifty years ago the opinions began to obtain that, in the teaching of mathematics, scholarship was an entirely secondary matter; that, in fact, any one could teach mathematics. At the present day this same pernicious opinion is still almost universal, and it is only in a comparatively small number of our colleges and scientific schools, and in a still smaller number of our secondary schools that the dangerous absurdity of this proposition has been recognized, and earnest attempts have been made to place the control of this branch in the hands of those properly fitted to take charge of it. Since the beginnings of education in this country, mathematics has been weighted down by the dead load of inefficient instruction so ignorantly fastened upon it. Under that load it is still weighted down. A long, hard struggle must be passed through before the subject is emancipated from its present burden; before the reform so earnestly desired and so acutely needed can be established.

For the study of mathematics four principal reasons may be given: 1. It is in itself a pure science and, like any other pure science, is worthy of study for its own sake. 2. It is invaluable as a means of mental discipline. 3. It is capable of innumerable practical applications. 4. It is necessary for the completion of all or nearly all prescribed school curricula. Of course it need not be said that the number of those who are influenced only by the first two, or either of the first two reasons is, in this country, wholly insignificant. Now and then a student is found who, like Rittenhouse, seems to have been actuated only by the purest love

for mathematics; or like Abraham Lincoln, who forsook his law studies for a time and devoted himself to Euclid for the sake of the mental drill and the training in rigorous accuracy of reasoning to be derived from geometry. But all such cases are to be considered merely as exceptions to the otherwise universal rule that students follow mathematics either because it is a part of their regularly prescribed course, or because it is in some way to be of direct practical benefit to them. In saying this I do not intend to say that all students dislike mathematics. On the contrary, experience shows that a fair proportion of the students of any class selected at random have a natural liking and aptitude for this work. In the preceding, reference is made to those who devote themselves to mathematics for either or both of the first two reasons to the exclusion of the last two. For such students instruction is almost unnecessary. They need now and then a suggestion from an experienced and widely read mathematician or logician—not daily recitations in the ordinary sense of the word. In fact, daily recitations as conducted by the ordinary teacher are precisely what such students do not need; for by that means errors in reasoning and loose habits of thought are inculcated which are sometimes detrimental in the highest degree to the student. From our present discussion, then, we eliminate these two classes, and proceed to consider mathematical instruction only as it affects those who are pursuing this subject with direct practical ends in view, and those who find in mathematics one element of the general education for which they are striving.

Wherein is the mathematical instruction of the present day lacking? Or, to ask the same question in a different manner, what are the particular points that will ensure good instruction in mathematics? Certain important but general truths may be stated in answer to this question, but they are truths that will apply to any other branch equally well. There are, however, a few elements of this question to be considered which are either peculiarly true of mathematics and mathematical teaching, or are of such transcendent importance in the general theory of teaching that they must not be lost sight of in our attempt to answer the question we have propounded.

At every step of our progress in mathematics, review is necessary. The truths of mathematics are, as has already been said, essentially abstract in their nature and are not easily fastened on

the mind. The common processes employed are forgotten by the ordinary student with a rapidity that would be discouraging to the teacher, were it not for his recollection of the ease with which the same things slipped out of his own mind but a few years before. Nothing but repetition, practice, practice, repetition; nothing but a constant, faithful, and long continued hammering can fix the common processes of mathematics in the mind so that their use will become, as it should be, automatic. To a certain extent this matter of review will regulate itself, so constantly does each successive stage of mathematical work involve what has gone before. But to a much greater extent this does not regulate itself; on the contrary, it requires on the part of the instructor unremitting vigilance to ensure a class against losing one week what was learned the week before. In using the word review, it must not be understood that reference is made to turning back and going over for a second time any part of a text-book. What is meant, and this cannot be too strongly urged, is that the handling of any subject should be made to involve to a greater or less degree as many as possible of the subjects already mastered. Text-books are often of but little assistance in this matter, and are sometimes a direct hindrance. Our text-books in algebra, geometry, and trigonometry are often the merest compilations, different bits of subject-matter "whacked" together by some ignoramus, ambitious to see his own name on a title page. Even our best books are far from perfect in this respect; and the teacher who would secure the highest degree of excellence from his students must at the outset make up his mind to be, in some degree at least, independent of his text-book.

No teacher of mathematics ever yet succeeded by trying to make the work too easy for his pupils. One of the most overfed, overgrown, and mischievous educational fads of the present day is the fad which has for its object the attempt to make school work in all its form a pleasant recreation to the learner. Study is not a recreation, and to be productive of good results it never can be a recreation. Study is work, and mathematical study is hard work. Any teacher who tries to persuade either himself or his class that this is not so, is unwise and dishonest. Mathematical work may be and should be made interesting, but it cannot be made play. Mental strength and mental accuracy can not be gained from algebra or geometry if the teacher insist upon mak-

ing of it a mere holiday excursion with himself as conductor. Inaccuracy and the tendency to shirk, which every teacher meets, must be ruled out with the most remorseless rigor. Leniency and infinite patience are necessary in guiding the first steps of a child in its primary work, but the high school and college are not the places for them. If the student is ever to walk alone, he must learn to do it there. But, it is asked, how about those who have for mathematics no natural aptitude whatever? How can such methods be applied, in any fairness, to them? I answer unhesitatingly, Do not compel them to study mathematics at all, beyond the necessary arithmetic of the grammar school and the merest elements of algebra and geometry. Mathematics can play no more part in the education of such minds than pig iron in the art of dressmaking. A part of the somewhat savage ardor that has in the past few years been displayed in attacking Greek, might not unwisely have been turned against the compulsory study of higher mathematics as well. The science of mathematics is the oldest and the noblest of all the sciences. Indeed it is the mother of many sciences. For the part it has played in the scientific and intellectual development of the world it is deserving of the profoundest veneration. But the proposition that higher mathematics, or even the whole of the ordinary mathematics of secondary schools, must constitute an essential part of the education of every man and woman who pursues study above the limits of the grammar grade is utter folly. The idea that no education can be worthy of the name unless it embraces trigonometry, analytical geometry, and calculus is the idea of those who do not know the meaning of the word education. And it is a significant fact that mathematicians rarely if ever hold to that idea. The time is not far distant when mathematics will assume in the curricula of secondary and higher schools its proper place; and when that time comes, mathematics will no longer be a bugbear in the school life of so many boys and girls. Allusion has been made to the difficulty, which so constantly confronts the student, of seeing what possible application can ever be made of many parts of his mathematical work. For this difficulty there is but one remedy; the teacher should be familiar enough with the subject of mathematics as a whole; should be enough of a mathematician to be able to point out at every step the relation between

the work immediately at hand and the work still to come. No matter how admirable a teacher he may otherwise be, he is lacking in one of the potent elements which go to constitute the success of the best mathematical instructors, if he knows nothing of his subject beyond the limits within which his own work of instruction lies. The most difficult parts of a student's work are oftenest those which seem to him to have no relation to anything else in the universe, past, present or future; which seem in short to be mere mathematical puzzles. But interest can be given to any and all such portions of the study, if the instructor only knows enough of his subject to enable him to do it. The same thing almost may be said with respect to the introduction of practical applications into daily class room work. The inability of an ordinary class in geometry, for example, to solve simple problems involving areas or solid contents is appalling. Ninety-nine teachers out of every hundred seem to feel that their duty has been fully done, if the subject matter of the text-book has been faithfully "gone over." Let their classes be tested by an examination consisting entirely of practical applications of that subject-matter. One such experience will take more conceit out of a teacher than a dozen teachers' conventions can put into his head. We hear much at the present day about the absurdity of teaching grammar through a series of years, only to have the student unable to write a simple letter correctly at the close of his course. Is it not equally absurd to ignore the practical side of mathematical work? The fact that text-books are different in this respect is no excuse for any teacher. The old question immediately comes up again, "Is the teacher to be master of his subject, or is he to be the slave of his text-book?"

Last of all, the teacher should shrink from no amount of personal labor in connection with his daily work. The preparation of problems, the correction of written work, and the many other direct and incidental demands upon his time combine into a serious burden; one which teachers would be glad to escape. But the teacher who allows himself to do this is unfaithful to his trust, and has his own ease more at heart than he has the interests of his class. Not only is there no royal road to mathematics, but there is no royal road to success in mathematical instruction.

*Levi L. Conant.*

*Worcester Polytechnic Institute.*